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A Multigenerational Game Model to Analyze Sustainable Development

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Abstract. This paper deals with a multigeneration game that provides a new rationale for representing time preference in very long term cost benefit analysis, as it happens typically in the economics of global climate change. One defines an intergenerational game where each generation has a random life duration and transfers the control of the economic system to the next generation at the end of its life. The payoff to a generation is a discounted sum of the expected consumption by the whole infinite sequence of generations, starting with the current one. The equilibrium is characterized by a dynamic programming equation; a unique solution is proved to exist; a numerical technique is proposed and implemented on a continuous time simplified version of the model DICE94. The results show the influence of this form of altruism on the asymptotic steady states of the economy subject to a global climate change effect.

Keywords: discounting, intergenerational equity, stochastic game, intergenerational equilibrium, turnpike solutions, cost-benefit analysis, climate change policies

1. Introduction

There is a growing consensus about the anthropogenic global climate change (GCC) induced mainly by the emissions of greenhouse gases (GHG) due to fossil fuel energy uses, industry and agriculture activities. To cope with anthropogenic GCC economies can rely on abatement, mitigation and adaptation. An interesting aspect of the problem is that abatement decided by the current generation, if any, will be made for the benefit of generations in a rather distant future and for the benefit of populations that are not currently the principal emitters. More generally the evaluation of sustainable development policies poses an ethical challenge addressed in Beltratti, Chichilnisky, and Heal (1995) and Chichilnisky (1996, 1997) through an axiomatic approach. In this paper we focus on the operational aspect of the problem: how to perform cost benefit analysis (CBA) for very long lived projects that overlap several generations? Among the proposals made by economists to deal with this issue one may distinguish (i) the recommendation to use a 0 discount (or pure time preference) rate when dealing with economic problem that involve many generations (F. Ramsey proposed this rule in 1928 (Ramsey, 1928) when he first studied the optimal economic growth problem), (ii) the use of a time dependent discount rate that would tend to 0 when $t \to \infty$ (Ainslie and Haslam, 1992; Harvey, 1994;

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Dasgupta, Mäler, and Barrett, 1999; Weitzman, 1998; 1999), and also a multigenerational game theoretic approach (Arrow, 1999) based on an idea first proposed in a different context in Phelps and Polak (1968). It has been argued that imposing a 0 discount rate imposes too much a toll on the current generation (Nordhaus, 1999; Tóth, 2000) and it has been shown that time dependent discount rates lead to time inconsistency. It is therefore the game theoretic approach that seems to offer the best way to design a time consistent evaluation system that realizes an equilibrium between the well being of current and future generations.

In this paper we further develop an intergenerational game model that has been first proposed in Haurie (to appear) as a way to introduce a form of altruism in long term CBA, without incurring the curse of time inconsistency. In the formalism of Haurie (to appear), each generation was interested in its own expected consumption and the one of the immediately following generation. That was a formalism close to the overlapping generation model in economic growth (Croix and Michel, 2002). In the present work we propose a multigeneration game model where each generation considers in its payoff the whole sequence of the expected rewards to all the future generations. A coefficient of altruism is introduced which plays a role similar to a discount factor applied to the rank of generation rather than to time (the weight given to a generation decreases with its rank).

As in Haurie (to appear) we introduce in Section 2 a piecewise deterministic control formalism where the control is exerted by a succession of players, also called generations, each having a random life duration. A general definition of an intergenerational equilibrium is given and we focus the rest of the paper on the case of a stationary control system where each generation has an exponentially distributed life duration that is introduced in Section 3. We then recall that such systems are likely to have steady-state attractors (also called turnpikes) for the equilibrium trajectory. We can then compare the turnpikes under different assumptions concerning the pure time preference (killing rate) or altruism parameters. Implementing a numerical technique that is explained in Haurie (to appear) we can show in Section 4 the impact of introducing this type of game theoretic formalism in a typical CBA applied to GCC, namely a simplified version of the DICE model (Nordhaus, 1994). In conclusion we draw some consequences for the conduct of CBA in the context of sustainable development.

2. A piecewise deterministic multigeneration game model

In this section we introduce a general dynamic game formalism that combines a piecewise deterministic control structure as e.g. in Boukas, Haurie, and Michel (1990) with a multi-generation stochastic game approach as proposed in Alj and Haurie (1983). We shall see that in the case where the randomness is described through exponentially distributed random life duration of each generation, this model generalizes the usual optimization of the discounted sum of rewards that is used in standard economic growth models.

2.1. A general formulation

Consider a dynamical system defined by the state equations

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{1}$$

$$u(t) \in U(t, x(t)) \tag{2}$$

$$x(0) = x^o. \tag{3}$$

A reward rate $L(t, x(t), u(t))$ is associated with the state and control variables. The system is controlled by a succession of players, each one representing a generation denoted $k \in \mathbb{N}$. The life duration of generation k is a random variable θ_k with finite expected value.

The state of the system is represented by $s = (t, \zeta, x)$, where $t \geq 0$, $\zeta \in \mathbb{N}$ and $x \in X \subset \mathbb{R}^n$. When the game begins, at $t^1 = 0$, generation 1 is in control. After a random life time θ_1 generation 1 dies and leaves control to generation 2, etc... If generation ℓ controls the system between time t^ℓ and time $t^\ell + \theta_\ell$ its *sum of rewards* is the integral $\int_{t^\ell}^{t^\ell+\theta_\ell} L(t, x(t), u(t))\, dt$. We shall assume that the payoff to any generation is constructed from the expected sums of rewards of all the forthcoming generations in a manner described below.

Definition 1. A strategy for generation k is a mapping from $[0, \infty) \times X$ into the class of mappings $w(\cdot) : [0, \infty) \to U$. If generation k takes control at time t^k, with initial state $x^k = x(t^k)$, then its control will be $u(t) = w(t - t^k)$ where $w(\cdot) = \gamma[t^k, x^k]$. This control generates a trajectory $x(\cdot) : [t^k, \infty) \to X$, with $x(t^k) = x^k$. The control generated is admissible as it satisfies $u(t) \in U(x(t))$.

Definition 2. The payoff to generation k, when it takes control at time $t^k = \sum_{\ell=1}^{k-1} \theta_\ell$, with state x^k is defined as

$$V_k[s^k; \gamma_k, \tilde{\gamma}_{k^+}] = \sum_{\ell \geq k} \alpha(k, \ell) \mathrm{E}_{\gamma_\ell} \left[\int_{t^\ell}^{t^\ell+\theta_\ell} L(t, x(t), u(t))\, dt | x^\ell = x(t^\ell) \right]. \tag{4}$$

In the above expressions the state trajectory $x(\cdot)$ is a solution of (1)-(3) with initial state x^k at initial time t^k.

Remark 3. The parameter $\alpha(k, \ell)$ defines the weight that generation k is placing on the rewards of generation ℓ when it computes its payoff. In Haurie (to appear) these coefficients are such that $\alpha(k, \ell) = 0$ if $\ell > k + 1$. Each generation was uniquely interested in the well being of the next generation. In the present paper we concentrate on the case where $\alpha(k, \ell) = \beta^{\ell-k}$, for $\ell \geq k$. When $\beta = 0$ the present generation is totally selfish; whereas, when $\beta = 1$ the present generation will consider, on an equal footing the well being of all the future generations.

Definition 4. An intergenerational equilibrium is a sequence $\underline{\gamma}^* = (\gamma_k^* : k \in \mathbb{N})$ such that,

$$\forall k \in \mathbb{N}, \quad \forall s^k \in S, \quad V_k(s^k; \gamma_k^*, \gamma_{k^+}^*) = \max_{\gamma_k} V_k(s^k; \gamma_k, \tilde{\gamma}_{k^+}^*). \tag{5}$$

Remark 5. In the above definition, each generation replies optimally to the expected behavior of all the forthcoming other generations. The influence a generation has on the rewards accrued to the forthcoming generations is essentially through the state value $x(t^k + \theta_k) = x^{k+1}$ which prevails when the control is transferred to the next generation.

In the above model we have a piecewise deterministic structure where the only randomness comes from the dates at which the control passes from one generation to the next. Between these random times the system behaves deterministically.

2.2. *Time consistency*

As it is now well established, this equilibrium concept is time consistent. It means that what is used by the current generation for computing its payoff, based on the control to be implemented by the coming generations, is consistent with the optimizing behavior of these generations when they will face the same problem. The lack of time consistency is the major drawback for the implementation of time dependent discount (time preference) rates. Intergenerational equilibria provide time consistency, usually at the expense of Pareto optimality as noticed already in Phelps and Polak (1968). However, in this particular case studied here, when the life duration is an exponential random variable and the weighting takes the simple form $\alpha(k, \ell) = \beta^{(\ell - k)}$, with $\beta < 1$, we obtain the rare situation where an equilibrium coincides with a Pareto or non-dominated solution.

2.3. *The case of exponentially distributed random life durations*

Assume that the life duration of generation ℓ is an exponential random variable with expectation $\frac{1}{\rho_\ell}$. Then, for generation ℓ, once it has adopted a control $u(\cdot) : [t^\ell, \infty) \to U(t, x(t))$, the expected reward sum can be expressed as

$$\mathrm{E}_{\gamma_\ell}\left[\int_{t^\ell}^{t^\ell + \theta_\ell} L(t, x(t), u(t))\, dt\right] = \int_0^\infty e^{-\rho_\ell t} L(t, x(t), u(t))\, dt \tag{6}$$

which is exactly the discounted sum of rewards over the infinite time horizon.[1] This reminds the well known fact that the discount rate ρ_ℓ (often also called the *killing rate*) is directly associated with the uncertainty about the life duration period. Therefore, in the case of exponentially distributed life durations, the above models leads to payoffs that are constructed from a sequence of infinite horizon discounted sums of rewards.

2.4. The limit case when there is no altruism

When there is no altruism ($\alpha(k,k) = 1$, $\alpha(k,\ell) = 0$ if $\ell > k$), each generation k, observing the inherited state x^k at time t^k when it receives control optimizes its expected sum of rewards. In the case of exponentially distributed life durations, it leads to a sequence of disconnected deterministic control problems with a performance criterion which is the discounted sum of rewards.

2.5. The limit case with total altruism

When there is full altruism ($\alpha(k,\ell) \equiv 1$), the problem should be equivalent to the case of a single infinitely lived generation, that is, the deterministic problem, with infinite horizon and without discounting.

3. The stationary exponential case

In this section one assumes that the life duration of generation k is an exponentially distributed random variable θ_k with expected value $\frac{1}{\rho}$. One also assumes that the controlled dynamical system is stationary

$$\dot{x}(t) = \tilde{f}(x(t), u(t)) \tag{7}$$

$$u(t) \in \tilde{U}(x(t)) \tag{8}$$

$$x(0) = x^o. \tag{9}$$

Assumption 1. The class of admissible controls maintains $x(t) \in X, \forall t \geq 0$, where X is a compact set in $\mathbb{R}^n$.

As in Haurie (to appear) we shall concentrate on the analysis of the attractors that can exist for the optimal trajectory obtained from the solution of this game.

3.1. Stationary dynamic programming equations

Because of time stationarity, the problem will have the same structure for each generation and one can define a generic generational payoff as

$$\tilde{V}(x^o; \gamma, \tilde{\gamma}') = \sum_{\ell=0}^{\infty} \beta^\ell \mathrm{E}_{\gamma^\ell}\left[\int_0^\theta \tilde{L}(x(t), u(t))\, dt\right], \tag{10}$$

where γ, resp. $\tilde{\gamma}'$, is the current generation (resp. next generations) strategy (resp. strategies), θ_0 (resp. θ^ℓ) is the life duration of present (resp. ℓ-th) generation, x^o is the initial state and $x(\cdot)$ is the trajectory, solution to the state equations (7)–(9) induced by these strategies. A stationary equilibrium is thus defined as a strategy γ that satisfies

$$\forall x \in X, \quad \tilde{V}^*(x) \equiv \tilde{V}(x;\gamma^*,\tilde{\gamma}^*) = \max_{\gamma} \tilde{V}(x;\gamma,\tilde{\gamma}^*). \tag{11}$$

Theorem 6. A stationary intergenerational equilibrium is characterized by the following dynamic programming equation:

$$\tilde{V}^*(x) = \max_{\gamma} \mathrm{E}_{\gamma}\left[\int_0^{\theta} \tilde{L}(x(t),u(t))\,dt + \beta\tilde{V}^*(x(\theta))|x(0)=x\right] \quad x \in X, \tag{12}$$

with the optimal strategy of each generation being obtained as the solution of the family of infinite horizon optimal control problems

$$\gamma^*(x) = \mathrm{argmax}_{u(\cdot)}\left[\int_0^{\infty} e^{-\rho t}(\tilde{L}(x(t),u(t)) + \rho\beta\tilde{V}^*(x(t)))\,dt|x(0)=x\right] x \in X. \tag{13}$$

Proof. By definition one has

$$\tilde{V}^*(x^o) = \sum_{\ell=0}^{\infty} \beta^{\ell}\mathrm{E}_{\gamma^*}\left[\int_{t^{\ell}}^{t^{\ell}+\theta_{\ell}} \tilde{L}(x(t),u(t))\,dt|x(t^{\ell})\right]. \tag{14}$$

Expressing the conditional expectation after the first random time θ one readily obtains (12). Since the elementary probability of the interval $[\theta, \theta + d\theta)$ is given by $\rho e^{-\rho\theta}\,d\theta$, and using equation (10) one can write

$$\tilde{V}(x^o;\gamma^*,\tilde{\gamma}^*) = \int_0^{\infty}\left(\int_0^{\theta} \tilde{L}(x^*(t),u^*(t))\,dt + \beta\tilde{V}^*(x^*(\theta))\right)\rho e^{-\rho\theta}\,d\theta,$$

where $x^*(\cdot)$ is the trajectory generated at initial state x^o by the control $u^*(\cdot)$ defined by the strategy γ^*. Integrate by parts the first integral to finally obtain

$$\tilde{V}^*(x^o) = \int_0^{\infty} e^{-\rho t}(\tilde{L}(x^*(t),u^*(t)) + \rho\beta\tilde{V}^*(x^*(t)))\,dt. \tag{15}$$

It suffices to write the equilibrium conditions to obtain the conditions (13). $\square$

Remark 7. The dynamic programming equation (12) is very similar to the one of a discrete event Markov Decision Process. This will help to prove existence of the equilibrium.

Theorem 8. For $\beta < 1$ the dynamic programming equation (12) admits a unique solution.

Proof. Let $\mathcal{V}$ be the Banach space of the Lipshitz continuous functions $\tilde{V}(x),\ x \in X$, endowed with the norm $\|\tilde{V}\| = \sup_{x\in X} |\tilde{V}(x)|$. Define the operator $\mathcal{T}^* : \mathcal{V} \to \mathcal{V}$ by

$$\mathcal{T}^*(\tilde{V})(x) = \max_{\gamma} \mathrm{E}_{\gamma}\left[\int_0^{\theta} \tilde{L}(x(t), u(t))dt + \beta \tilde{V}(x(\theta))|x(0) = x\right] \quad \tilde{V} \in \mathcal{V} \quad x \in X \quad (16)$$

where $u(\cdot)$ and $x(\cdot)$ are the solution to (7) and (8) with initial condition $x(0) = x$ induced by the strategy γ. It is an easy, matter (see Appendix 1), using standard results of the theory of Markov decision processes (see e.g. (Puterman, 1994)) to prove that the operator $\mathcal{T}^*$ is contracting for all $\beta < 1$, hence the result. □

3.2. *The equilibrium is also Pareto optimal*

Consider the problem consisting of optimizing the discounted sum over all generations of their expected rewards

$$\max_{\{\gamma^{\ell}\}_{\ell=1\ldots\infty}} \tilde{V}(x^o; \{\gamma^{\ell}\}_{\ell=1\ldots\infty}) = \sum_{\ell=0}^{\infty} \beta^{\ell} \mathrm{E}_{\gamma^{\ell}}\left[\int_0^{\theta} \tilde{L}(x(t), u(t))\, dt\right]. \quad (17)$$

It is a simple matter to write the dynamic programming equation for this problem and to find that it also leads to (12) which characterizes the intergenerational equilibrium. The link between intergenerational equilibrium and the dynamic programming conditions in stochastic control models has already been exposed in Alj and Haurie (1983).

We can give an interesting interpretation of this property. The model proposed has two intertwined discounting schemes. The first one is associated with the random life duration of each generation, the second one is not associated with time per se but, instead, with the rank of the coming generations.[2]

Another interesting aspect of this piecewise deterministic control system is that the random times when generations switch from one to the next in control are not influenced by the control. Furthermore, the only event that takes place at these random times is a renewal of the control which passes to a new player. As we have seen, the successive generations will use the same strategy and, therefore the trajectory will not be affected by these random switches. As we shall see in more details in the next subsection, this trajectory will be the solution of an associated auxiliary deterministic and implicit infinite horizon control problem.

3.3. *The associated implicit infinite horizon control problem*

Theorem 6 characterizes the equilibrium strategy of a generation as the solution of a family of auxiliary infinite horizon ρ-discounted optimal control problems having an implicitly defined reward function

$$\mathcal{L}^*(x(t), u(t)) = \tilde{L}(x(t), u(t)) + \rho\beta\tilde{V}^*(x(t)). \tag{18}$$

We call it "implicit" because the function $\tilde{V}^*(x(t))$ is itself defined by the very strategy γ^* characterized by the solution of this control problem. The auxiliary infinite horizon control problem is defined as

$$\max \int_0^\infty e^{-\rho t}\mathcal{L}^*(x(t), u(t))\, dt \tag{19}$$

s.t.

$$\dot{x}(t) = \tilde{f}(x(t), u(t)) \tag{20}$$

$$u(t) \in \tilde{U}(x(t)) \tag{21}$$

$$x(0) = x^o. \tag{22}$$

3.4. *Long run steady state analysis*

Under sufficient concavity and curvature assumptions (see Carlson, Haurie, and Leizarowitz (1994) for details) the infinite horizon optimal control problem (19)–(22) may have an attractor $\bar{x}$ common to all trajectories, emanating from different initial states x^o. This attractor is a trajectory steady-state that solves the following implicit programming problem (Feinstein and Luenberger, 1981):

$$\max \ \mathcal{L}^*(x, u) = L(x, u) + \rho\beta\tilde{V}^*(x) \tag{23}$$

s.t.

$$0 = f(x, u) + \rho(x - \bar{x}) \tag{24}$$

$$u \in U(x) \tag{25}$$

where $\bar{x}$ is the turnpike itself. When the altruism parameter β tends to 0, the above turnpike will tend to coincide with the one associated with the usual discounted reward optimal control problem.

$$\max \ L(x, u) \tag{26}$$

s.t.

$$0 = f(x, u) + \rho(x - \bar{x}) \tag{27}$$

$$u \in U(x) \tag{28}$$

So, in terms of asymptotic behavior of the optimal trajectories, altruism is modifying the attractor by replacing the optimized reward $L(x, u)$ with a modified reward $\mathcal{L}(x, u) = L(x, u) + \rho\beta\tilde{V}^*(x)$. We can do static comparative analysis to assess the impact of this modification on the asymptotic steady-state.

When $\rho \to 0$ one may expect the expression $\rho\, \tilde{V}^*(x)$ to tend toward $\frac{g^*}{1-\beta}$ where g^* is the maximal sustainable reward, solution of

$$g^* = \max L(x, u) \tag{29}$$

s.t.

$$0 = f(x, u) \tag{30}$$

$$u \in U(x). \tag{31}$$

Therefore, the limit steady-state problem will be

$$\max \ \mathcal{L}^*(x, u) = L(x, u) + \frac{g^*}{1-\beta} \tag{32}$$

s.t.

$$0 = f(x, u) \tag{33}$$

$$u \in U(x). \tag{34}$$

Clearly this problem admits the same solution as the maximal sustainable reward one defined in (26)–(28). So, when the discount (killing) rate $\rho \to 0$ the optimal trajectory is similar to the one associated with the infinite horizon, undiscounted control problem.

3.5. Numerical technique to compute a solution

The numerical computation of the fixed point of the operator $\mathcal{T}^*$ defined in (16) can be undertaken through an adaptation of the Kushner-Dupuis method (Kushner and Dupuis, 1992).

To implement that method we represent the optimal strategy as a feedback loop and we implement a continuous time dynamic programming approach. Assuming regularity for this value function $\tilde{V}^*(x)$ and using standard dynamic programming arguments one can characterize the value function as the solution to the Hamilton-Jacoby-Bellman equation

$$\rho \tilde{V}^*(x) = \max_{u \in \tilde{U}(x)} \left\{ \tilde{L}(x, u) + \rho\beta \tilde{V}^*(x) + \frac{\partial}{\partial x} \tilde{V}^*(x) \tilde{f}(x, u) \right\}. \tag{35}$$

In equation (35), one approximates the partial derivative $\frac{\partial}{\partial x_i} \tilde{V}^*(x)$ by finite differences taken in the direction of the flow, that is:

$$\frac{\partial}{\partial x_i} \tilde{V}^*(x) \to \begin{cases} (\tilde{V}^*(x + e_i h) - \tilde{V}^*(x))/h & \text{if } \tilde{f}_i(x, u) \geq 0 \\ (\tilde{V}^*(x) - \tilde{V}^*(x - e_i h))/h & \text{if } \tilde{f}_i(x, u) < 0 \end{cases} \tag{36}$$

where e_i is the unit vector of the i-th axis. Define

$$\tilde{f}_i^+(x, u) = \max\{0, \tilde{f}_i(x, u)\}$$
$$\tilde{f}_i^-(x, u) = \max\{0, -\tilde{f}_i(x, u)\}.$$

Substituting the differences to the partial derivatives in equation (35), one gets

$$\rho \tilde{V}^*(x) = \max_{u \in \tilde{U}(x)} \left\{ \tilde{L}(x,u) + \rho\beta \tilde{V}^*(x) + \sum_{i=1}^{n} \left(\frac{(\tilde{V}^*(x + e_i h) - \tilde{V}^*(x))}{h} \tilde{f}_i^+(x,u) \right.\right.$$
$$\left.\left. - \frac{(\tilde{V}^*(x) - \tilde{V}^*(x - e_i h))}{h} \tilde{f}_i^-(x,u) \right) \right\}. \tag{37}$$

which yields

$$0 = \max_{u \in \tilde{U}(x)} \left\{ h\tilde{L}(x,u) - \tilde{V}^*(x) \left(\rho\,(1-\beta)h + \sum_{i=1}^{n} \left(\tilde{f}_i^+(x,u) + \tilde{f}_i^-(x,u) \right) \right) \right.$$
$$\left. + \sum_{i=1}^{n} (\tilde{f}_i^+(x,u)\tilde{V}^*(x + e_i h) + \tilde{f}_i^-(x,u)\tilde{V}^*(x - e_i h)) \right\}. \tag{38}$$

Define the interpolation interval

$$\Delta_h = \frac{h}{\rho\, h + \sum_{i=1}^{n} |f_i(x,u)|}. \tag{39}$$

Where we have used the fact that $\tilde{f}_i^+(x,u) + \tilde{f}_i^-(x,u)) = |f_i(x,u)|$. One considers an MDP with discrete states x^g, $g \in \mathcal{G}$ and control $u \in \tilde{U}(x^g)$. The transition rewards are given by $(\tilde{L}(x^g,u) + \rho\, \beta\, \tilde{V}^*(x))\Delta_h$. The transition probabilities $\Pi_i(x^g, x^{g'}, u)$ are defined as follows:

- When $g \in \mathcal{G} \setminus \partial\mathcal{G}$ the transition probabilities x^g to any neighboring sampled value $x^g \pm e_i h$ are given by

$$\pi_i^{\pm}(x^g,u) = \frac{\tilde{f}_i^{\pm}(x^g,u)}{\sum_{i=1}^{n} |f_i(x,u)| + \rho\, \beta\, h}.$$

- A renewal jump occurs with probability

$$\pi_{\mathrm{ren}}(x^g,u) = \frac{\rho\, \beta\, h}{\sum_{i=1}^{n} |f_i(x,u)| + \rho\, \beta\, h}.$$

- On the boundary $\partial\mathcal{G}$ of the grid, the probabilities are defined according to a reflecting boundary scheme.
- All the other transition probabilities are 0.

A discounting term is defined by

$$\beta(x^g,u) = \frac{\sum_{i=1}^{n} |f_i(x^g,u)| + \rho\, \beta\, h}{\sum_{i=1}^{n} |f_i(x^g,u)| + \rho\, h}.$$

The DP equations for this approximating MDP are given by

$$v(x^g) = \max_{u \in \tilde{U}(x^g)} \Bigg\{ \tilde{L}(x^g, u)\Delta_h + \beta(x^g, u)(\pi_{\text{ren}}(x^g, u)v(x^g) + \sum_{i=1,\dots,n} \sum_{\pm} \pi_i^{\pm}(x^g, u)v(x^g \pm he_i) \Bigg\} \tag{40}$$

that we also write more concisely

$$v(x^g) = \max_{u \in \tilde{U}(x^g)} \Bigg\{ \tilde{L}(x^g, u)\Delta_h + \beta(x^g, u) \sum_{g' \in \mathcal{G}} \Pi_i(x^g, x^{g'}, u)v(x^{g'}) \Bigg\} \tag{41}$$

where $\Pi_i(x^g, x^{g'}, u)$ stands for the controlled transition probabilities defined above. Expliciting these probabilities we obtain

$$v(x^g) = \max_{u \in \tilde{U}(x^g)} \Bigg\{ \Delta_h \Bigg(\tilde{L}(x^g, u) + \beta\rho v(x^g) + \sum_{i=1}^{n} (\tilde{f}_i^{+}(x^g, u)v(x^g + e_i h) + \tilde{f}_i^{-}(x^g, u)v(x^g - e_i h)) \Bigg) \Bigg\} \tag{42}$$

We solve this MDP using linear programming (see Puterman (1994)), after having also discretized the control variables (see Appendix 2 for details).

4. Application to GCC integrated assessment

In this section we use the simplified continuous time version of the DICE model (Nordhaus, 1994, 1999) already introduced in Haurie (to appear), to illustrate the impact of multigeneration equilibrium approach on GCC integrated assessment.

4.1. A reduced model based on DICE94

We shall illustrate this approach on the following, very simplified, dynamical system structure representing an economy with a single homogenous good that can be consumed or invested to obtain productive capital; the output depends on two production factors, labor and capital. Table 1 gives the full list of variables entering the model.

$$\max_{c(\cdot)} \quad \int_0^\infty e^{-\rho t} U(c(t), L(t))\, dt \tag{43}$$

$$U(c(t), L(t)) = L(t) \frac{c(t)^{1-\alpha} - 1}{1 - \alpha} \tag{44}$$

$$\dot{L}(t) = g_L(t) L(t) \tag{45}$$

Table 1
List of variables in the DICE94 model.

		List of endogenous state variables
$K(t)$	=	capital stock
$M(t)$	=	mass of GHG in the atmosphere
$T(t)$	=	atmospheric temperature relative to base period
$T^*(t)$	=	deep-ocean temperature relative to base period
		List of control variables
$I(t)$	=	gross investment
$\mu(t)$	=	rate of GHG emissions reduction
		List of exogenous dynamic variables
$A(t)$	=	level of technology
$L(t)$	=	labor input (=population)
$O(t)$	=	forcing exogenous GHG
		List of auxiliary variables
$C(t)$	=	total consumption
$c(t)$	=	per capita consumption
$D(t)$	=	damage from GH warming
$E(t)$	=	emissions of GHGs
$F(t)$	=	radiative forcing from GHGs
$\Omega(t)$	=	output scaling factor due to emissions control and to damages from climate change
$Q(t)$	=	gross world product

$$\dot{g}_L(t) = -\delta_L g_L(t) \tag{46}$$

$$Q(t) = \Omega(t)A(t)K(t)^\gamma L(t)^{1-\gamma} \tag{47}$$

$$\dot{A}(t) = g_A(t)A(t) \tag{48}$$

$$\dot{g}_A(t) = -\delta_A g_A(t) \tag{49}$$

$$Q(t) = C(t) + I(t) \tag{50}$$

$$c(t) = \frac{C(t)}{L(t)} \tag{51}$$

$$\dot{K}(t) = I(t) - \delta K(t) \tag{52}$$

$$E(t) = (1 - \mu(t))\sigma(t)Q(t) \tag{53}$$

$$\dot{M}(t) = \beta_0 E(t) - \delta_M(M(t) - 590) \tag{54}$$

$$F(t) = 4.1\frac{\log[M(t)] - \log[590]]}{\log[2]} + O(t) \tag{55}$$

$$\dot{T}(t) = \frac{1}{R_1}\{F(t) - \lambda T(t) - \frac{R_2}{\tau_{12}}\{T(t) - T^*(t)\} \tag{56}$$

$$\dot{T}^*(t) = \frac{1}{\tau_{12}}\{T(t) - T^*(t)\} \tag{57}$$

$$D(t) = Q(t)\theta_1(T(t) + \theta_2 T(t)^2) \tag{58}$$

$$TC(t) = Q(t)b_1\mu(t)^{b_2} \tag{59}$$

$$\Omega(t) = \frac{1 - b_1 \mu^{b_2}}{1 + \theta_1 (T + \theta_2 T^2)}. \tag{60}$$

This model is basically an economic growth model linked with a global temperature model. A damage function computes the output loss due temperature increase and abatement control. The model is well adapted to CBA analysis at a global level and has become a landmark in the domain of global climate change integrated assessment.

4.2. Numerical analysis of asymptotics

We report now the result of numerical approximations for the identification of the asymptotic attractors, or "turnpikes" under different equilibrium specifications.

4.2.1. Without altruism

This system is not time homogenous because of population growth and technical progress. However the model implies asymptotic values for these exogenous variables that are given below

$$\begin{aligned} \bar{A} &= 0.063 \\ \bar{L} &= 12000 \\ \bar{O} &= 1.15 \end{aligned}$$

We therefore use these asymptotic values in our computations of a stationary intergenerational equilibrium. The Table 3 gives the asymptotic steady state of any trajectory for $\rho = 0\%$ and $\rho = 6\%$, respectively. We notice the important effect of the pure rate of time preference on the asymptotic environmental state, represented by the variables T and M. Also, discounting implies a lower level of asymptotic consumption.

4.2.2. With altruism

We now compute the asymptotic steady states for trajectories that satisfy the intergenerational equilibrium introduced in this paper. Using the method described in Appendix 3 we obtain the figures of Table 4 showing the asymptotic attractors from $\beta = .1$, $\beta = .5$ and

Table 2
Parameter values.

α	=	1
b_1	=	0.045
b_2	=	2.15
β_0	=	0.64
γ	=	0.25
δ_K	=	0.10 (per year)
δ_M	=	0.0833 (per decade)
λ	=	1.41
θ_1	=	0.0007
θ_2	=	3.57
σ	=	0.033

Table 3
Turnpike values when $\rho = 0\%$ and $\rho = 6\%$, respectively.

State variables	State variables
$\bar{K} = 943$	$\bar{K} = 506$
$\bar{M} = 911$	$\bar{M} = 1170$
$\bar{T} = 2.64$	$\bar{T} = 3.69$
Control variables	*Control variables*
$\bar{I} = 94.3$	$\bar{I} = 50.6$
$\bar{\mu} = 0.68$	$\bar{\mu} = 0.36$
Exogenous variables	*Exogenous variables*
$\bar{A} = 0.063$	$\bar{A} = 0.063$
$\bar{L} = 12000$	$\bar{L} = 12000$
$\bar{O} = 1.15$	$\bar{O} = 1.15$
Auxiliary variables	*Auxiliary variables*
$\bar{C} = 291$	$\bar{C} = 278$
$\bar{E} = 41.83$	$\bar{E} = 75.47$
$\bar{F} = 3.72$	$\bar{F} = 5.20$
$\bar{Q} = 400$	$\bar{Q} = 342$
$\rho = 0\%$	$\rho = 6\%$

Table 4
Turnpike values when $\rho = 6\%$ and $\beta = 0.1, 0.5, 0.8, 0.99$, respectively.

State variables	*State variables*	*State variables*	*State variables*
$\bar{K} = 530$	$\bar{K} = 663$	$\bar{K} = 810$	$\bar{K} = 871$
$\bar{M} = 1151$	$\bar{M} = 1077$	$\bar{M} = 1010$	$\bar{M} = 955$
$\bar{T} = 3.62$	$\bar{T} = 3.34$	$\bar{T} = 3.07$	$\bar{T} = 2.83$
Control variables	*Control variables*	*Control variables*	*Control variables*
$\bar{I} = 53$	$\bar{I} = 66$	$\bar{I} = 81$	$\bar{I} = 87.14$
$\bar{\mu} = 0.36$	$\bar{\mu} = 0.47$	$\bar{\mu} = 0.57$	$\bar{\mu} = 0.63$
Exogenous variables	*Exogenous variables*	*Exogenous variables*	*Exogenous variables*
$\bar{A} = 0.063$	$\bar{A} = 0.063$	$\bar{A} = 0.063$	$\bar{A} = 0.063$
$\bar{L} = 12000$	$\bar{L} = 12000$	$\bar{L} = 12000$	$\bar{L} = 12000$
$\bar{O} = 1.15$	$\bar{O} = 1.15$	$\bar{O} = 1.15$	$\bar{O} = 1.15$
Auxiliary variables	*Auxiliary variables*	*Auxiliary variables*	*Auxiliary variables*
$\bar{C} = 280$	$\bar{C} = 286$	$\bar{C} = 290$	$\bar{C} = 290$
$\bar{E} = 73$	$\bar{E} = 63.42$	$\bar{E} = 54.62$	$\bar{E} = 47.48$
$\bar{F} = 5.10$	$\bar{F} = 4.71$	$\bar{F} = 4.33$	$\bar{F} = 4$
$\bar{Q} = 347$	$\bar{Q} = 366$	$\bar{Q} = 385$	$\bar{Q} = 392$
$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.80$	$\beta = 0.99$

$\beta = .99$, respectively, with still a pure time preference of 0.06 % within the life span of each generation. We notice that, as expected, the introduction of altruism transforms the asymptotic steady state into a solution which lies half-way between the discounted-no-altruism solution ($\rho = 0.06$ and $\beta = 0$) and the undiscounted solution ($\rho = 0$). When β increases, as observed previously, the asymptotic steady-state gets closer to the solution of the undiscounted single generation problem. The introduction of altruism increases the environmental concern, favors higher capital accumulation and reduces emissions through a more active abatement policy.

5. Conclusion

The intergenerational equilibrium that has been characterized in this paper offers a rationale for proposing criteria to be used in CBA when the projects span several generations. The whole approach boils down finally to a modification of the reward (utility) function entering the economic growth model, by valuing the stocks of physical and environmental capital at each instant of time. In the approach developed here, this valuation is often practised by economists of the environment.[3] This value is obtained endogenously from an altruistic consideration of the wellbeing of forthcoming generations. Using known results of the theory of asymptotic control and numerical methods borrowed from stochastic control theory, we have analyzed the effect of this form of altruism on the attracting steady states (turnpikes) of the growing economy. These effects correspond to the expected consequences of intergenerational altruism in the treatment of long term environmental problems, like, typically, the anthropogenic global climate change. The scheme proposed in this paper is linked with the growing literature on intergenerational equilibria (see Sorger, 2002); using a double discounting process, one intra-generational described by the pure time preference rate ρ, and one inter-generational given by the altruism parameter β, we have obtained a formulation which reduces to an optimization scheme. This is contrasting with Haurie (to appear) where the altruism is fully concentrated on the next generation in row. In the scheme proposed in Haurie (to appear), time consistency is indeed obtained but not Pareto optimality. An open domain of investigation is the experimentation of these approaches in dynamic economic models that are not time homogenous.

Appendix 1: Proof of contraction property for operator $\mathcal{T}^*$

We consider two functions $\tilde{V}$ and $\tilde{W}$ in $\mathcal{V}$. $\mathcal{T}^*$ maps $\mathcal{V}$ into $\mathcal{V}$. Fix $x \in X$, assume $\mathcal{T}^*(\tilde{V})(x) \geq \mathcal{T}^*(\tilde{W})(x)$ and let

$$\gamma^* = \mathrm{argmax}_{\gamma} \left\{ \mathrm{E}_{\gamma} \left[\int_0^{\theta} \tilde{L}(x(t), u(t))\, dt + \beta \tilde{V}(x(\theta)) | x(0) = x \right] \right\}.$$

Then

$$\begin{aligned}0 \leq \mathcal{T}^*(\tilde{V})(x) - \mathcal{T}^*(\tilde{W})(x) &\leq \mathrm{E}_{\gamma^*}\left[\int_0^\theta \tilde{L}(x(t), u(t))\,dt + \beta \tilde{V}(x(\theta))|x(0) = x\right] \\ &\quad - \mathrm{E}_{\gamma^*}\left[\int_0^\theta \tilde{L}(x(t), u(t))\,dt + \beta \tilde{W}(x(\theta))|x(0) = x\right] \\ &= \beta \int_0^\infty \rho\, e^{-\rho t}(\tilde{V}(x^*(t)) - \tilde{W}(x^*(t)))\,dt \\ &\leq \beta\,\|\tilde{V} - \tilde{W}\|.\end{aligned}$$

We repeat this argument in the case that $\mathcal{T}^*(\tilde{V})(x) \leq \mathcal{T}^*(\tilde{W})(x)$ to get finally

$$|\mathcal{T}^*(\tilde{V})(x) - \mathcal{T}^*(\tilde{W})(x)| \leq \beta\,\|\tilde{V} - \tilde{W}\|, \quad \forall x \in X \tag{61}$$

and thus, taking the supremum w.r.t. $x \in X$ we get the desired result. □

Appendix 2: The linear program used to solve the associated MDP

We introduce the variables $z^g = v(x^g)$. Let Ξ be a grid for the control set $\tilde{U} = \bigcup_{x \in X} \tilde{U}(x)$ and denote u^ξ, $\xi \in \Xi$, the sampled control points. For each x^g, $g \in \mathcal{G}$, we denote $\Xi^g = \{\xi \in \Xi : u^\xi \in \tilde{U}(x^g)\}$. We then solve the following LP

$$\begin{aligned}&\text{minimize } \sum_{g \in \mathcal{G}} z^g \\ &\qquad \text{s.t. } z^g \geq \tilde{L}(x^g, u^\xi)\Delta_h + \beta(x^g, u^\xi) \sum_{g' \in \mathcal{G}} \Pi_i(x^g, x^{g'}, u) z^{g'}, \quad g \in \mathcal{G},\ \xi \in \Xi.\end{aligned}$$

This is a very large LP indeed. With $n = 3$, $m = 2$ and 10 discretization points on each dimension, we end up with 1000 variables and 100’000 constraints. These LP are solved easily (less than 2 minutes) on a 700 Mhz processor PC, using XPress. AMPL (Fourer, Gay, and Kernighan, 1993) was used to submit the model to the optimizer.

Appendix 3: The numerical computation scheme

Once the intergenerational game is solved, the associated LP gives the values $z^g = v(x^g)$ for the discretization points. Typically we use a grid in 3 dimensions with 10 points on each axis; so we end up with 1000 values, corresponding to different sampled values for the state variables K, M and T. We then adjust a polynomial regression to these data, obtaining very good R^2 values. This polynomial function is used as an approximation of the function $\tilde{V}^*(x)$. We then proceed to solve the implicit programming problem:

$$\max \quad \mathcal{L}^*(x, u) = L(x, u) + \rho\beta\tilde{V}^*(x) \tag{62}$$

s.t.

$$0 = f(x, u) + \rho(x - \bar{x}) \tag{63}$$

$$u \in U(x). \tag{64}$$

We do it by implementing a Gauss-Seidel method where the values $\bar{x}$ are updated sequentially as given by the solution of the problem (63)-(64), with a fixed $\bar{x}$.

Notes

1. Recall that an exponential random variable T with parameter ρ_ℓ has the half line $[0, \infty)$ as support with a distribution function $\mathrm{P}[T \leq \theta] = 1 - e^{-\rho_\ell \theta}$. The expected value of T is then $\frac{1}{\rho}$. Now, since the elementary probability of T being in the interval $\theta, \theta + d\theta$ is given by $\rho e^{-\rho_\ell \theta}\, d\theta$, we can write

$$\mathrm{E}_{\gamma_\ell}\left[\int_{t^\ell}^{t^\ell+\theta_\ell} L(t, x(t), u(t))\, dt\right] = \int_0^\infty \left(\int_0^\theta L(t, x(t), u(t))\, dt\right)\rho e^{-\rho_\ell \theta}\, d\theta.$$

   It suffices to integrate by parts to obtain the result (6) that is the discounted value of the infinite stream of rewards.
2. As is natural, we may have a decreasing interest in what happens to our grand, grand...grand children.
3. See Ambrosi et al. (to appear) for an interesting analysis of the different elements that enter into the utility function for integrated assessment models.

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